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# ON THE RECOGNITION COMPLEXITY OF SOME GRAPH PROPERTIES

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By applying a topological approach due to Kahn, Saks and Sturtevant, we prove that all decreasing graph properties consisting of bipartite graphs only are elusive. This is an analogue to a well-known result of Yao.

## 1. Introduction

Let $T$ denote a finite set and $\mathcal{P}$ a property of the subsets of $T$ by which we mean that $\mathcal{P} \subset 2^T$, the power set of $T$. Suppose that two players $\mathcal{A}$ (*Algy*) and $\mathcal{S}$ (*Strategist*) are playing the following game: Player $\mathcal{A}$ wants to learn from $\mathcal{S}$ whether some unknown set $X \subset T$ is in $\mathcal{P}$ or not. For each $x \in T$, he is allowed to ask the question *Is* $x \in X$?

Player $\mathcal{A}$ wants to minimize the number of questions but $\mathcal{S}$ provides answers in order to force $\mathcal{A}$ to ask as many questions as possible. If both players play optimally from their point of view, then the number of questions which are asked in the game is called the *recognition complexity* of $\mathcal{P}$ (also: *Boolean decision tree complexity*) and is denoted by $c(\mathcal{P})$. If $\mathcal{S}$ can force $\mathcal{A}$ to probe all elements of $T$, then $\mathcal{P}$ is called *elusive* (also: *evasive*).

As an example for a successful strategy of $\mathcal{S}$, suppose that $|\mathcal{P}|$ is odd. A question divides $2^T$ and thus $\mathcal{P}$ into two disjoint parts, namely the sets in $\mathcal{P}$ for which the answer is yes and those for which it is no. Exactly one of the parts has odd cardinality. Imagine $\mathcal{S}$ to give his answers such that at each time the number of sets in $\mathcal{P}$ which are compatible with all the answers given is odd. Then after $|T|-1$ questions, two possible sets in $2^T$ remain and exactly one of the two sets is in $\mathcal{P}$. It follows that the last question must be posed and hence that $\mathcal{P}$ is elusive.

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The recognition complexity has been studied by various authors during the last twenty years (see all the references except [11], [14] and [15]). Most of the efforts were concentrated on the case that $T$ equals $V^{(2)}$, the set of two-element subsets of some finite set $V$. In this case the subsets of $T$ can be interpreted as graphs with vertex set $V$. A *graph property* is a subset $\mathcal{P} \subset 2^T$, $T = V^{(2)}$, such that $\mathcal{P}$ contains with each graph $G$ also each isomorphic copy of $G$ (with vertex set $V$). A graph property $\mathcal{P}$ is called *nontrivial* if $\mathcal{P} \neq \emptyset, \mathcal{P} \neq 2^T$. It is called *decreasing* if, for each $G \in \mathcal{P}$, all subgraphs of $G$ (with vertex set $V$) are contained in $\mathcal{P}$, *increasing* if $2^T \backslash \mathcal{P}$ is decreasing and *monotonic* if it is increasing or decreasing. The research began in 1973, when Rosenberg [12] conjectured that there exists some $\gamma > 0$ such that

$$c(\mathcal{P}) \geq \gamma n^2$$

for all nontrivial graph properties $\mathcal{P}$, $n = |V|$.

This original conjecture soon turned out to be false (for counterexamples see [2]), but the counterexamples suggested the following modified version which Rosenberg formulated together with Aanderaa:

*There exists $\gamma > 0$ such that for all nontrivial, monotonic graph properties $\mathcal{P}$ we have $c(\mathcal{P}) \geq \gamma n^2$.*

The modified conjecture was proved by Rivest and Vuillemin in 1975 with $\gamma = 1/16$ (see [13]). Kleitman and Kwiatkowski [8] improved the value of $\gamma$ from $1/16$ to $1/9$ for $n$ large. A breakthrough came in 1984, when Kahn, Saks and Sturtevant [9] applied methods from algebraic topology to prove that $c(\mathcal{P}) \geq n^2/4 + o(n^2)$. In fact, they proved the following conjecture of Karp if $n$ is a prime power:

*All nontrivial, monotonic graph properties are elusive.*

Karp's conjecture is still open for all $n \geq 10$ which are not prime powers. The new method was applied by King [7] to digraph properties and by Yao [19] to monotone bipartite graph properties. More precisely, Yao showed that nontrivial monotone properties of bipartite graphs *with two given color classes* are elusive. The main result of the present paper is the analogue of Yao's theorem in the original model, namely: *All decreasing graph properties consisting of bipartite graphs only are elusive.* For some other applications of the topological method in the original model, see [18]. In the following we use the graph theoretical notions from [4]. All the prerequisites about permutation groups can easily be found in [10]. We also assume familiarity with the most fundamental notions of homology theory. Here the standard reference is [15].

## 2. Some results from algebraic topology

We assume from now on that $\mathcal{P}$ is a monotone, nontrivial graph property.

Since $c(\mathcal{P}) = c(2^T \setminus \mathcal{P})$, it suffices to study decreasing graph properties which can be seen as simplicial complexes. Hence, the *Euler characteristic* $\chi(\mathcal{P})$ of $\mathcal{P}$ is defined as

$$\chi(\mathcal{P}) := \sum_{i=1}^{|T|}(-1)^{i-1}a_i$$

where $a_i := a_i(\mathcal{P}) := |\{X \in \mathcal{P} : |X| = i\}|$.

Furthermore, for each abelian group $G$ (here: $G=\mathbb{Z}$ or $G=\mathbb{Z}/p\mathbb{Z}$, $p$ prime), we may consider the homology groups with coefficients in $G$. The complex $\mathcal{P}$ is called *$G$-acyclic* if

$$H_0(\mathcal{P}, G) = G \quad \text{and} \quad H_i(\mathcal{P}, G) = 0 \quad \text{for } i > 0$$

where $H_i(\mathcal{P}, G)$ denotes the $i$-dimensional homology group of $\mathcal{P}$ with respect to $G$. The following result due to Kahn, Saks and Sturtevant is the bridge between complexity theory and algebraic topology we need:

**Theorem 1.** [Kahn, Saks and Sturtevant] *If $\mathcal{P}$ is not elusive, then it is contractible and hence $\mathbb{Z}_p$-acyclic for all primes $p$.*

Suppose now that $G$ is a permutation group acting on $T$ and leaving $\mathcal{P}$ invariant. We define the simplicial complex $\mathcal{P}_G$ with vertex set $T_G := \{B \in \mathcal{P} : B \text{ orbit of } G\}$ by

$$\{B_1, \ldots, B_k\} \in \mathcal{P}_G :\Longleftrightarrow B_1 \cup \ldots \cup B_k \in \mathcal{P}.$$

The interesting fact is that, for special permutation groups $G$, homological properties of $\mathcal{P}$ are reflected by homological properties of $\mathcal{P}_G$. More precisely, consider the set $\mathcal{G}$ of all finite groups $G$ containing a normal $p$-subgroup $G'$, $G' \trianglelefteq G$, such that the factor group $G/G'$ is cyclic. P. A. Smith and R. G. Oliver (see [14] and [11]) proved the following result:

**Theorem 2.** [Smith, Oliver] *If $G \in \mathcal{G}$ is a permutation group on $T$ leaving the $\mathbb{Z}_p$-acyclic complex $\mathcal{P}$ invariant, then*

$$\chi(\mathcal{P}_G) = 1.$$

*In particular, $\mathcal{P}_G$ is nonempty.*

## 3. The Sylow $p$-subgroups of $S_n$ and their invariant graphs

The results of the previous section suggest that it might be worthwile to look at the Sylow $p$-subgroups of the symmetric group $S_n \simeq S_V$ as well as the cyclic group $C_n$ acting on the set $V^{(2)}$ of all edges. For the applications of this paper it would be sufficient to discuss the case $p=2$. Since it is not more difficult, we give, however the discussion for general $p$. Before starting with the details, recall the following two definitions from the theory of permutation groups:

Suppose $Q$ and $R$ are permutation groups acting on sets $U$ and $W$, respectively. Then the *wreath product* $G := Q \wr R$ is defined as

$$Q \wr R := \{(f;\rho): \ f: W \to Q, \ \rho \in R\},$$

with the following action on $U \times W$:

$$(f;\rho)(u,w) := (f(\rho(w))(u), \rho(w)).$$

$Q$ and $R$ are called *similar* if there is a bijection $f: U \to W$ and an isomorphism $\phi: Q \to R$ such that for all $u \in U$ and $\pi \in Q$ the following holds:

$$(\phi(\pi))(f(u)) = f(\pi(u)).$$

The Sylow $p$-groups of $S_n$ have been investigated by Kaloujnine [6]. We are now going to summarize the relevant results (all the proofs can be found in [6]): Suppose that $n = \sum_{i \in I} a_i p^i$ is the (unique) representation of $n$ in the p-adic number system ($a_i \geq 1$ for $i \in I$). We choose a partition of $V$ into pairwise disjoint sets $M_{i,j}$

$$V = \bigcup_{i \in I} \bigcup_{j=1}^{a_i} M_{i,j}, \ |M_{i,j}| = p^i, 1 \leq j \leq a_i, \ i \in I,$$

and Sylow $p$-subgroups $Q_{i,j}$ of $S_{M_{i,j}} \simeq S_{p^i}$. Then the direct sum

$$\bigoplus_{i \in I} \bigoplus_{j=1}^{a_i} Q_{i,j}$$

is a Sylow $p$-subgroup of $S_n$ and, up to similarity, we can obtain all Sylow $p$-subgroups in this way.

Hence it suffices to consider the case that $n$ is a prime power, $n = p^m$. Assume further that $V$ is the vector space $\mathbb{Z}_p^m$. In order to construct a Sylow $p$-subgroup $Q_m$ of $S_n$ we consider functions $a(x_1, \ldots, x_s): \mathbb{Z}_p^s \to \mathbb{Z}_p$, $s = 0, \ldots, m-1$. For each such system of functions we define $m$ permutations $\pi(a)$, $\pi(a(x_1))$, $\ldots$, $\pi(a(x_1, \ldots, x_{m-1}))$ on $\mathbb{Z}_p^m$ as follows: For $y := (y_1, \ldots, y_m) \in \mathbb{Z}_p^m$ we let

$$\pi(a(x_1, \ldots, x_{s-1}))(y) := (y_1, \ldots, y_{s-1}, y_s + a(y_1, \ldots, y_{s-1}), y_{s+1}, \ldots, y_m).$$

We denote the product $\pi(a(x_1, \ldots, x_{m-1})) \cdots \pi(a(x_1))\pi(a)$ of these permutations (in $S_n$) by the $m$-tuple $A := (a, a(x_1), \ldots, a(x_1, \ldots, x_{m-1}))$ which we call (with Kaloujnine) a *tableau*. The functions $[A]_s := a(x_1, \ldots, x_{s-1})$ are called the *coordinates* of $A$. The set of all permutations which can be represented by tableaux as above form a Sylow $p$-subgroup $Q_m$ of $S_n$. We identify $Q_m$ with the group of all tableaux where the product of two tableaux $A$ and $B = (b, b(x_1), \ldots, b(x_1, \ldots, x_{m-1}))$ is of course defined as the tableau representing the product of the permutations which are represented by $A$ and $B$. We have the following equations (for $1 \leq s \leq m$):

$$[AB]_s = a(x_1, \ldots, x_{s-1}) + b(x_1 - a, \ldots, x_{s-1} - a(x_1, \ldots, x_{s-2}))$$

and

$$[A^{-1}]_s = -a(x_1 + a, x_2 + a(x_1 + a), \ldots, x_{s-1} + a(x_1 + a, x_2 + a(x_1 + a), \ldots)).$$

For recursions, it is more convenient to rewrite the equations as follows: Suppose that $X_{m-1} := (x_1, \ldots, x_{m-1}) \in \mathbb{Z}_p^{m-1}$ and

$$A_{m-1} := (a, a(x_1), \ldots, a(x_1, \ldots, x_{m-2})) \in Q_{m-1}.$$

Let $X_{m-1}^{A_{m-1}} := (x_1 - a, \ldots, x_{m-1} - a(x_1, \ldots, x_{m-2}))$ and define $B_{m-1}$ and $X_{m-1}^{B_{m-1}}$ analogously. Then the above equations can be written as

$$\begin{gathered}(A_{m-1}, a(X_{m-1}))(B_{m-1}, b(X_{m-1})) = \\ \left(A_{m-1}B_{m-1}, a(X_{m-1} + b\left(X_{m-1}^{A_{m-1}}\right)\right)\end{gathered}$$

and

$$(A_{m-1}, a(X_{m-1}))^{-1} = \left(A_{m-1}^{-1}, -a\left(X_{m-1}^{A_{m-1}^{-1}}\right)\right).$$

The group $Q_m$ is isomorphic to the $m$-fold wreath product of the cyclic group $C_p$ with itself and acts transitively on $\mathbb{Z}_p^m$. All Sylow $p$-subgroups of $S_{p^m}$ are similar to $Q_m$.

We are now able to determine the orbits of $Q_m$ acting on $V^{(2)} = \left(\mathbb{Z}_p^m\right)^{(2)}$.

**Theorem 3.**

(i) *For* $p=2$ *and* $0 \le i \le m-1$ *let* $B_i^m := B_i^m(p) := \{\{(X,x,Y),(X,x',Y')\} \mid X \in \mathbb{Z}_p^{m-i-1},\ x,x' \in \mathbb{Z}_p, x \neq x',\ Y,Y' \in \mathbb{Z}_p^i\}$. *Then the* $B_i^m$ *are just the orbits of* $Q_m$ *acting on* $V^{(2)}$.

(ii) *For* $p > 2$ *and* $0 \le i \le m-1$, $1 \le j \le (p-1)/2$ *let* $B_{i,j}^m := B_{i,j}^m(p) := \{\{(X,x,Y),(X,x',Y')\} \mid X \in \mathbb{Z}_p^{m-i-1},\ x,x' \in \mathbb{Z}_p,\ x - x' \in \{-j,j\},\ Y,Y' \in \mathbb{Z}_p^i\}$. *Then the* $B_{i,j}^m$ *are just the orbits of* $Q_m$ *acting on* $V^{(2)}$.

**Proof.** We proceed by induction on $m$ and treat the cases (i) and (ii) simultaneously:

The case $m=1$ is easy to check directly. Since $Q_1$ is cyclic, it is also possible to apply Proposition 3 (see below). The details are left to the reader.

Now suppose that $m>1$ and that the theorem is proved for $m-1$. We first consider edges of the form $\{(X_{m-1}, x_m), (X_{m-1}, x'_m)\}$ with $X_{m-1} \in \mathbb{Z}_p^{m-1}$, $x_m, x'_m \in \mathbb{Z}_p$, $x_m \neq x'_m$. Since $Q_{m-1}$ acts transitively on $\mathbb{Z}_p^{m-1}$, the orbit of such an edge consists of all edges $\{(Y_{m-1}, y_m), (Y_{m-1}, y'_m)\}$ with $Y_{m-1} \in \mathbb{Z}_p^{m-1}$ and $y_m, y'_m \in \mathbb{Z}_p$, $y_m - y'_m = \pm(x_m - x'_m)$ (or just $y_m \neq y'_m$for $p=2$). It follows immediately that the orbits of those edges are just the sets $B_{0,j}^m$, $1 \le j \le (p-1)/2$, for odd $p$ and $B_0^m$ for $p=2$.

Now let $X_{m-1}, X'_{m-1} \in \mathbb{Z}_p^{m-1}$, $X_{m-1} \neq X'_{m-1}$, $x_m, x'_m \in \mathbb{Z}_p$, $A \in Q_m$. We have:

$$A\Big\{\big(X_{m-1}, x_m\big), \big(X'_{m-1}, x'_m\big)\Big\} =$$

$$\Big\{\big(A_{m-1}X_{m-1}, x_m + a(A_{m-1}X_{m-1})\big), \big(A_{m-1}X'_{m-1}, x'_m + a(A_{m-1}X'_{m-1})\big)\Big\}.$$

Fix $A_{m-1} \in Q_{m-1}$ and imagine that the last coordinate $[A]_m$ varies over all functions from $\mathbb{Z}_p^{m-1}$ into $\mathbb{Z}_p$. Since $X_{m-1} \neq X'_{m-1}$, the pairs

$$\big([A_m](A_{m-1}X_{m-1}), [A_m](A_{m-1}X'_{m-1})\big) = \big(a(A_{m-1}X_{m-1}), a(A_{m-1}X'_{m-1})\big)$$

exhaust all pairs in $\mathbb{Z}_p \times \mathbb{Z}_p$. We conclude that $\{(Y_{m-1}, y_m), (Y'_{m-1}, y'_m)\}$ is in the same $Q_m$-orbit as $\{(X_{m-1}, x_m), (X'_{m-1}, x'_m)\}$ if and only if $\{X_{m-1}, X'_{m-1}\}$ is in the same $Q_{m-1}$-orbit as $\{Y_{m-1}, Y'_{m-1}\}$. Now the result follows immediately from the induction hypothesis and the following two equations which are easily checked:

$$B_{i,j}^m = \Big\{\{(X_{m-1}, x_m), (X'_{m-1}, x'_m)\} | \{X_{m-1}, X'_{m-1}\} \in B_{i-1,j}^{m-1}\Big\}$$

for $p > 2$ and $1 \le i \le m-1$, $1 \le j \le (p-1)/2$ as well as

$$B_i^m = \Big\{\{(X_{m-1}, x_m), (X'_{m-1}, x'_m)\} | \{X_{m-1}, X'_{m-1}\} \in B_{i-1}^{m-1}\Big\}$$

for $p = 2$ and $1 \le i \le m-1$. ∎

For $p = 2$, the theorem implies immediately the following proposition about the graphs $G_i^m = (V, B_i^m)$:

**Proposition 1.** *Every graph $G_i^m$ is the vertex-disjoint union of $2^{m-i-1}$ components, each one isomorphic to the complete bipartite graph $K_{2^i,2^i}$ $0 \le i \le m-1$. The components of $G_{i-1}^m$ are spanned by the color classes of the components of $G_i^m$, $1 \le i \le m-1$. In particular, the union of any two of the graphs $G_i^m$ contains a triangle.*

As a first application, we prove that for even $n$ the number of graphs on $V$ with a perfect matching is odd. It follows that the property of having a perfect matching is elusive, a result which was proved in a different way in [18].

**Proposition 2.** *If $n \ge 2$ is even, $n = |V|$, then the number of graphs on vertex set $V$ with a perfect matching is odd.*

**Proof.** Suppose that $V = \{1, \dots, n\}$ and that $Q$ is a Sylow 2-subgroup of $S_n$. Denote further by $\mathcal{P}$ the system of all graphs on $V$ with a perfect matching. The action of $Q$ on $\mathcal{P}$ partitions $\mathcal{P}$ into orbits whose lengths are powers of two, hence

$$|\mathcal{P}| \equiv |\{G \in \mathcal{P} : G \text{ is a } Q\text{-invariant graph on } V\}| \bmod 2.$$

We write $n=\sum_{i\in I} 2^i$ and partition $V$ into disjoint sets $M_i$ of cardinality $2^i$ $(i\in I)$. We further assume that $Q=\bigoplus_{i\in I} Q_i$, where $Q_i$ is a Sylow 2-subgroup of $S_{M_i}$. Since each $Q_i$ acts transitively on $M_i$, the $Q$-invariant graphs can all be constructed by first choosing $Q_i$-invariant graphs on $M_i$ for $i\in I$ and then for $i\neq j$ drawing either all edges between $M_i$ and $M_j$ or none.

*Claim:* A $Q$-invariant graph $G$ has a perfect matching if and only if each $M_i$ with $E(G[M_i])=\emptyset$ is joined to some $M_j$ with $j>i$.

To prove necessity, assume that $G$ has a perfect matching and does not contain edges with both endpoints in $M_i$. Since a perfect matching matches all points in $M_i$ with points outside $M_i$ and $\sum_{j<i} 2^j < 2^i$, some $M_i - M_j$-edge where $j>i$ must exist.

For the converse, observe that by Proposition 1, each nonempty $Q_i$-invariant graph on $M_i$ contains a perfect matching $F_i$. Fix such a matching $F_i$ for all $i\in I$ with $E(G[M_i])\neq\emptyset$. Then we successively choose edge sets $E_i\subset E(G)$ for the $i\in I$ with $E(G[M_i])=\emptyset$ with the following properties:

- There is some $j>i$ such that all the edges in $E_i$ join elements from $M_i$ to elements from $M_j$.
- If $xy$ is an edge from $F_j$ and $x$ endpoint of an $E_i$-edge, then the same holds for $y$.
- All edges in $\bigcup_{k\leq i} E_k$ are independent.

It is obvious that this is possible and that the edge set $\bigcup_i E_i$ can be enlarged to become a perfect matching of $G$ by adding edges from $\bigcup_j F_j$. The claim is proved.

The proposition now follows by induction on $n$: The case $n=2$ being trivial, we assume that $n\geq 4$ and that the proposition is true for smaller (even) cardinalities. If $n$ is a power of two, $n=2^m$ say, then it follows from the previous proposition that the $Q$-invariant graphs with a perfect matching are just all the $2^m-1$ unions $\bigcup_{j\in J} G_j$, where $\emptyset\neq J\subset\{0,\dots,m-1\}$, hence our result. If $n$ is not a power of two, let $i_0:=\min I$. By our claim, a $Q$-invariant graph $G$ has a perfect matching if and only if the induced subgraph $G[\bigcup_{i\in I\setminus\{i_0\}} M_i]$ has a perfect matching and exactly one of the following two conditions holds:

(a) $M_{i_0}$ is joined to some $M_i$, $i\in I\setminus\{i_0\}$, or

(b) $M_{i_0}$ is not joined to some $M_i$, $i\in I\setminus\{i_0\}$, but $G[M_{i_0}]$ has a perfect matching.

The number of $Q$-invariant graphs $G$ with a perfect matching and condition (a) is even, since in any such graph we may replace $G[M_{i_0}]$ by any one of the $2^{i_0}$ $Q_{i_0}$-invariant graphs without affecting condition (a). The number of $Q$-invariant graphs with a perfect matching and condition (b) equals the product

$$|\{H : H \text{ is a } Q_{i_0}\text{-invariant graph with a perfect matching on } M_{i_0}\}|\times$$
$$|\{H : H \text{ is } \textstyle\bigoplus_{j\in I\setminus\{i_0\}} Q_j\text{-invariant with a perfect matching on } \bigcup_{j\in I\setminus\{i_0\}} M_j\}|.$$

The first factor is $\equiv 2^{i_0}-1 \equiv 1 \bmod 2$, the second is odd by the induction hypothesis. Summarizing, we have shown that $|\mathcal{P}| \equiv 0+1\cdot 1 \equiv 1 \bmod 2$ and the proof is complete. ∎

It would be interesting to have an elementary proof of Proposition 2 which does not use results on Sylow 2-subgroups. For our main theorem, we also need (part of) the following proposition which can be found in [18] and is very easy to prove:

**Proposition 3.** *Let $V := \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ and $\tau : \mathbb{Z}_n \to \mathbb{Z}_n$ where $\tau(x) := x+1$. The group $G := \langle \tau \rangle$ generated by $\tau$ acts canonically on $V^{(2)}$ and the following statements about the orbits of $G$ hold:*

(i) *If $n$ is odd, $V^{(2)}$ decomposes into $(n-1)/2$ orbits of cardinality $n$.*

(ii) *If $n$ is even, $V^{(2)}$ decomposes into $\frac{n}{2}-1$ orbits of cardinality $n$ and one orbit of cardinality $n/2$.*

(iii) *The set of edges $\{\{0,j\} : 1 \le j \le \lfloor n/2 \rfloor\}$ is a system of distinct representatives for the orbits of $G$.*

(iv) *Denote by $D_j^n = D_j$ the orbit of $\{0,j\}$ and by $F_j^n = F_j$ the corresponding graph $(V, D_j)$. Then $F_j$ is the vertex disjoint union of $s_j$ cycles of length $r_j$, where $s_j := \gcd(j,n)$ and $r_j := n/s_j$, $(1 \le j < n/2)$. For $n$ even we have $F_{n/2} \simeq \frac{n}{2}K_2$. The components of $F_j$ are the sets $M_{j,i}^n := M_{j,i} := \{x \in \mathbb{Z}_n : x \equiv i \bmod s_j\}$, $0 \le i < s_j, 1 \le j \le \lfloor n/2 \rfloor$.*

## 4. Decreasing graph properties consisting of bipartite graphs only

We are now going to apply the topological approach to prove our main result.

**Theorem 4.** *All (nontrivial) decreasing graph properties consisting of bipartite graphs only are elusive.*

**Proof.** Let $\mathcal{P}$ denote such a graph property on $V$ and suppose first that $n$ is odd. Consider the transitive action of a cyclic group $G$ on $V$. By Proposition 3, all the orbits of $G$ on $V^{(2)}$ contain cycles whose lengths divide $n$ and are thus odd. It follows that $\mathcal{P}_G$ is empty. By Theorems 1 and 2, $\mathcal{P}$ is elusive.

So assume that $n$ is even and $x \in V$. Consider now a cyclic group $G$ acting transitively on $V \setminus \{x\}$. Arguing as above we see that a non-elusive $\mathcal{P}$ must contain the $G$-orbit consisting of all the edges incident to $x$. Hence we may assume that $\mathcal{P}$ contains the graph $K_{1,n-1}$.

Now suppose that player $\mathcal{A}$ probes an edge $xy$ and player $\mathcal{S}$ answers with "yes". We are left with the set of all graphs in $\mathcal{P}$ containing the edge $xy$. Let $\mathcal{P}'$ denote the set of all $B \subset V^{(2)} \setminus \{xy\}$ such that the graph $H = (V, B \cup \{xy\})$ is in $\mathcal{P}$. Then

$\mathcal{P}'$ is a (nontrivial) simplicial complex which is invariant under the induced action of the symmetric group on $V' := V \setminus \{x,y\}$. We are now going to choose a group $G \in \mathcal{G}$ acting transitively on $V'$ with the following property: All simplices in $\mathcal{P}'_G$ are zero-dimensional, i.e. they consist of one $G$-orbit only. If this can be done, then the proof is easily completed which can be seen as follows: Since $K_{1,n-1} \in \mathcal{P}$, $\mathcal{P}'_G$ contains at least the orbits $B_x := \{xz : z \in V'\}$ and $B_y := \{yz : z \in V'\}$. We infer that the Euler characteristic $\chi(\mathcal{P}'_G)$ is at least two. By Theorems 1 and 2, $\mathcal{P}'$ and thus also $\mathcal{P}$ are elusive.

To construct a suitable group $G$, write $|V'| =: n' = 2^m k$ with natural numbers $m$ and $k$ where $k$ is odd. W.l.o.g. we assume that $V' = U \times W$ with $U = \mathbb{Z}_2^m$. Let $Q$ denote a Sylow 2-subgroup of the symmetric group on $U$ as described above and $R$ a cyclic group acting transitively on $W$. Consider the wreath product $G := Q \wr R$ as defined in section 3. It is well-known that $G$ contains the normal subgroup $G' := \{(f;1_R) : f : W \to Q\}$ which is isomorphic to $Q^k$ and hence a 2-group. Since the factor group $G/G'$ is isomorphic to the cyclic group $R$, $G$ is in $\mathcal{G}$. By Theorem 3 and the definition of the wreath product, it is easy to see that the $G$-orbits of edges with both endpoints in some set $U \times \{w_0\}$ are just the sets $C_j := \{\{(u,w),(u',w)\} : \{u,u'\} \in B_j^m, w \in W\}, 0 \le j \le m-1$, where the $B_j^m$ are as in Theorem 3. In particular, the union of any two of the $C_j$ contains a triangle by Proposition 1 and thus cannot belong to $\mathcal{P}'$. Now assume that $D$ is the orbit of an edge $\{(u,w),(u',w')\}$ with $w \neq w'$. Then $D$ contains $\{(u,w),(u,w')\}$ as well and hence all the edges in $F := \{\{(u,\rho(w)),(u,\rho(w'))\} : \rho \in R\}$. But $F$ is an orbit of the cyclic group $R$ acting on $\{u\} \times W$ and therefore contains an odd cycle by Proposition 3. It follows that the vertices of the simplicial complex $\mathcal{P}'_G$ are among the orbits $B_x, B_y$ and $C_j$, $0 \le j \le m-1$. Since the union of any two of these orbits (together with the edge $xy$) contains a triangle, the complex $\mathcal{P}'_G$ is zero-dimensional. ∎

In [18] it was proved that decreasing graph properties whose graphs contain neither a triangle nor a $C_4$ are elusive. We pose the problem to prove the same result if $C_4$ is not excluded. The elusiveness of triangle-free decreasing graph properties would of course also generalize Theorem 4.

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